Brittle intergranular fracture of two-dimensional disordered solids as a random walk

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We present results of a hybrid experimental and numerical investigation of the fracture of two-dimensional materials with a granular structure. We characterize the statistical properties of a series of intergranular cracks in conditions where the characteristic scale of damage is much smaller than the grain size. We show that crack paths exhibit mono-affine Gaussian properties characterized by a roughness exponent $\zeta = 0.50 \pm 0.05$ that can be explained from linear elastic fracture mechanics calculations. Our findings support the description of the roughening process in two-dimensional brittle disordered solids by a directed random walk.

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Deciphering the statistical properties of fracture surfaces has been a long-standing goal in condensed matter physics [1, 2]. This has been driven both by curiosity and by the exploration of the microscopic failure mechanisms that govern the macroscopic resistance of materials. Fracture surfaces that reflect the complex interaction of cracks with microscopic material features represent indeed a ready-made pathway to explore these mechanisms. Following this idea, the scaling $\langle \delta h^2 \rangle \sim \delta r^{2\zeta}$ of crack roughness with the observation scale $\delta r$, where the exponent $\zeta$ takes universal values [3, 5], has raised hope that a unified and simple description of failure mechanisms may be achieved. A major question yet unanswered is the interpretation of this scaling behavior, and more specifically the value of the roughness exponent in terms of failure mechanisms.

An open problem is the gap between the persistent ($\zeta > 0.5$) fracture profiles measured in experiments on both two-dimensional (2D) thin sheets [6–10] and three-dimensional (3D) solids [1, 11, 12] and the anti-persistent crack paths ($\zeta \leq 0.5$) [13, 15] predicted using linear elastic fracture mechanics (LEFM) based theories. This paradox could be explained recently for 3D solids thanks to the observation of fracture surfaces with either anti-persistent ($\zeta_{3D} \approx 0.4$) or logarithmic correlations of heights ($\zeta_{3D} \approx 0$) [15, 16] in agreement with the LEFM prediction. In these experiments, the scaling properties of cracks were measured at large scales $\delta r \gg \ell_{pe}$ with respect to the characteristic size of the fracture process zone lying in the crack tip vicinity, thus respecting a key hypothesis of the LEFM theory [13, 18]. These findings also helped to resolve the paradox of the abnormally hight exponent $\zeta_{3D} \approx 0.75$ measured on metallic alloys [4], mortar [11], wood [20] or quasi-brittle rocks [12]. It was indeed realized that persistency was reminiscent of the small scale ($\delta r \ll \ell_{pe}$) behavior of fracture surfaces, and was signature of the damage mechanisms taking place within the fracture process zone. This picture was finally consolidated by the observation of two separate scaling regimes: (i) a damage-driven roughness at small scale with $\zeta_{3D} \approx 0.75$, and (ii) an LEFM-consistent roughness at larger scale, on the very same fracture surfaces of mortar [11], phase-separated glass [18], and finally on a large range of materials [21].

Surprisingly, such a level of understanding is far from being reached for the roughness of fractured 2D thin sheets. In particular, whether the LEFM theory can be helpful or not in this context is still an open question. The pseudo-line tension of the crack front resulting from bulk elasticity that was shown to dominate the roughening process in 3D elastic solids [13, 15, 22] does not play any role in 2D. Instead, the crack path is governed by the stress state near the tip which depends on the local crack inclination and the past trajectory [23, 24]. The latter effect tends to maintain the crack path as close as possible to a straight line. Consequently, LEFM based models of crack propagation in disordered 2D solids predict anti-persistent fracture profiles, with $\zeta_{2D} \leq 0.5$, or even no self-affine behavior [25, 24]. These predictions are in contradiction with experiments that systematically report exponents in the range $\zeta_{2D} \approx 0.6 – 0.7$ as in paper sheets [6, 8], wood [9] or metallic alloy [10]. Recent numerical works that take into account the nucleation, growth and coalescence of cavities during the process of crack propagation in 2D materials report values ($\zeta_{2D} = 0.65 – 0.70$) close to those obtained in experiments [27, 28]. These results also compare well with the predictions of random fuse and random beam models that describe qualitatively microcracking processes in quasi-brittle solids [29, 30]. If these findings suggest that the persistent self-affine regime reported in experiments...
emerges from damage and microcracking processes, the existence of a scale invariant regime of roughness in brittle 2D thin sheets is still a matter of debate \[25,26\]. The exploration of this regime on experimental and numerical examples is the central point of this Letter.

In the spirit of Refs. \[14,17\], we use 2D consolidated granular materials exhibiting intergranular failure as archetypes of disordered brittle materials. Crack paths are investigated experimentally on thin sheets made of polystyrene beads, and numerically in large scale 2D simulations of intergranular fracture of random arrangements of polygonal grains. The statistics of experimental and computed crack profiles is then fully characterized and shown to be reminiscent of the directed random walk with roughness exponent \( \zeta \approx 0.5 \). In the last part of this Letter, we take inspiration from Refs. \[25,26\] and propose an LEFM based model of crack propagation through disordered brittle solids that is later used to explain quantitatively our findings. Our study suggests an explanation for the apparent discrepancy between theory and experiments regarding the scaling of fracture paths in 2D disordered solids.

**Experimental fracture tests of two-dimensional granular solids** – To explore the crack morphology produced by brittle intergranular failure, we use commercial panels of expanded polystyrene. Each panel consists of consolidated pre-expanded polystyrene beads with an average radius \( \ell \approx 2 \text{ mm} \). The radius of the beads is comparable to the sample thickness, but is a few hundred times smaller than the other dimensions of the specimens. A notch is machined at the center of one of the short edges of the plate and serves as the location of crack initiation. The experiments were conducted at constant displacement rate through a four-point bending geometry (see Supplemental Material \[31\] for a detailed description of the experimental setup). The bead cohesion was sufficiently low to ensure brittle intergranular failure as confirmed by post mortem observations of the fracture surfaces. Crack profiles were extracted using digital image analysis of the broken sample photographs, an example of which is shown on Fig. 1(a).

**Simulations of intergranular failure** – Crack propagation in a material with a realistic microstructure is investigated by means of a numerical approach \[32,33\] that reproduces fairly well the processes into play in experiments, namely intergranular brittle failure of random packing of grains. A representative example of the Voronoi microstructures considered in this study, with 3140 Voronoi grains and about 320 broken grain boundaries in the process region, is reported in Fig. 1(b). These microstructures are embedded in the process region of a notched specimen. Although the numerical results have been obtained for a well defined polycrystalline material, they are expected to be representative of any consolidated granular material with zero porosity under the assumption that dissipative failure processes are confined to grain boundaries and embedded within a process zone of size \( \ell_{\text{px}} \) much smaller than the grain boundary length \( \ell \). Under these conditions, Shabir et al. \[33\] showed that intergranular crack paths are not influenced by cohesive law parameters. This implies that in the limit of brittle intergranular failure, there exists a unique crack path for a given microstructure, irrespective of cohesive property values and material type. The computational approach and the test setup are detailed in the Supplemental Material \[31\].

**Roughness characterization of the crack profiles** – The statistics of the experimental and computed crack profiles are now investigated. We start by computing their height-height correlation function defined as

\[
\Delta h(\delta x) = \langle [h(x + \delta x) - h(x)]^2 \rangle_{x}^{1/2}
\]

(a) Brittle intergranular crack profiles obtained with experimental fracture tests of thin sheets of expanded polystyrene. (b) Large scale simulations of cohesive fracture of random arrangements of polygonal elastic grains.

![FIG. 1. (a) Brittle intergranular crack profiles obtained with experimental fracture tests of thin sheets of expanded polystyrene. (b) Large scale simulations of cohesive fracture of random arrangements of polygonal elastic grains.](image-url)
This contrasts with crack paths in paper sheets that not only display larger exponents $\zeta \simeq 0.65 - 0.70$ [8], but also multi-scaling [31]. This comparison suggests that, similarly to recent observations in 3D solids [21, 35, 36], the mechanism of crack propagation leaves signature on the value of the roughness exponent and on the multi-scaling properties. For brittle materials, like the one investigated in this work, crack paths are mono-affine and close to a directed random walk ($\zeta \simeq 0.5$); while cracks propagating through damage nucleation, growth and coalescence like in paper sheets, leave persistent ($\zeta > 0.5$) fracture profiles with multi-scaling.

**Theoretical interpretation** – To test this scenario, we explore theoretically the statistical properties of a brittle crack propagating through a material with a disordered microstructure. Our analysis is based on the principle of local symmetry [23, 37] and assumes that the crack propagates along the direction of vanishing shearing mode II [38]. Following the idea of Katzav et al. [25, 26], we start from the calculation of Amestoy and Leblond [24] that predicts the propagation direction $\theta$ from the local value of the stress intensity factors as

$$
\theta(x^+) = -2\frac{k_{\text{II}}(x)}{k_1(x)}.
$$

We discretize the crack path $h(x)$ that is approximated by a succession of straight segments of extension $\Delta x$ along the mean propagation direction $x$ that, subsequently, will be taken in the limit $\Delta x \to 0$. As a result, the angle $\theta(x^+)$ in Eq. (2) provides the propagation direction on the right side of $x$ while the local stress intensity factors $k_1(x)$ and $k_{\text{II}}(x)$ are calculated from the crack path configuration before propagation. We limit our analysis to cracks with small slopes $\theta \ll 1$ as observed in simulations and experiments, so that $\theta(x^+) \simeq \frac{dh}{dx^+} - \frac{dh}{dx^x}$. In order to predict the crack trajectory, the local stress intensity factors must be expressed as a function of the crack geometry $h(x)$. We use the results of Cotterell and Rice [23], enriched by the work of Movchan et al. [39], that provides approximate expressions of $\{k_1^{\text{hom}}, k_{\text{II}}^{\text{hom}}\}$ as a function of $h(x)$, the macroscopic stress intensity factors $\{K_1^{(0)}, K_{\text{II}}^{(0)}\}$ and the coefficients $\{T^{(0)}, A_1^{(0)}\}$ of the higher order terms in the development of the stress field near the tip of a crack embedded in a homogeneous elastic medium:

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{k_1^{\text{hom}}(x)}{K_1^{(0)}} = 1 \\
\frac{k_{\text{II}}^{\text{hom}}(x)}{K_{\text{II}}^{(0)}} = 1 + \frac{K_1^{(0)}}{2} \frac{dh}{dx} + \sqrt{\frac{\pi}{2}} T^{(0)} \int_{-\infty}^{h'} \frac{h''(u)}{\sqrt{x-u}} du.
\end{array} \right.
\end{align*}
$$

Since in this study $K_{\text{II}}^{(0)} = 0$, inserting these expressions in Eq. (2) yields to the following closed form of the
profiles, the path equation \( \delta h = h'(x) dx \) is integrated numerically using either the values \( \{ L_1, L_2 \} \) calculated from the actual experimental loading conditions, or \( \{ L_1^{sim}, L_2^{sim} \} \) for the simulations. The height-height correlation function computed from the solution obtained using \( \{ L_1^{exp}, L_2^{exp} \} \) is represented in Fig. 4 and displays a random walk behavior with roughness exponent \( \zeta_{rw} = 1/2 \) for \( \delta x \gg \ell \). A similar behavior is obtained if \( \{ L_1^{sim}, L_2^{sim} \} \) are used instead.

The correlation of slopes can be calculated analytically as shown in the Supplementary Materials \([31]\). The terms inversely proportional to \( \sqrt{L_1} \) and \( L_2 \) in the path equation \( \delta h = h'(x) dx \) provide negligible contributions, thus leading to

\[
\langle h'(x + \delta x) h'(x) \rangle_x \sim \langle \eta(x + \delta x) \eta(x) \rangle_x \quad (6)
\]

This is confirmed by the direct numerical resolution of the path equation. The slope correlator obtained using \( \{ L_1^{exp}, L_2^{exp} \} \) is shown in the inset of Fig. 4 and indeed merges with the disorder correlator. This indicates that the predicted crack slopes display no correlations at scales \( \delta x \gg \ell \), similarly to the fracture profiles obtained through experiments and simulations.

Finally, the Gaussian behavior of the simulated fractures reported in Fig. 3 can be directly inferred from the absence of correlations of \( h'(x) \). Expressing the height variation at scale \( \delta x \) as the sum of \( n = \delta x/\ell \) independent variables \( \delta h = \int_x^{x+\delta x} h'(x) dx \approx \sum_{k=0}^{n-1} h'(x+kt) \) shows, using the central limit theorem, that the distributions \( P_{\delta h}(\delta h) \) follow a Gaussian statistics for \( \delta x \gg \ell \).

Discussion — Our theoretical analysis of the path followed by brittle cracks in 2D disordered solids captures quantitatively all the statistical features of the intergranular crack paths observed experimentally and numerically, namely (i) the random walk behavior with roughness exponent \( \zeta \approx 0.50 \), (ii) the absence of correlations between slopes along the crack path, and (iii) the Gaussian statistics of height fluctuations. A major assumption of the model, that is also a key feature of the experimentally and numerically fractured materials, is the clear scale separation between the grain size \( \ell \) that serves as an elementary step during crack growth, and the process zone size \( \ell_{pz} \), where damage and dissipative mechanisms take place in the crack tip vicinity. We believe that the comparison between these both length scales selects the toughening regime in 2D solids: (i) when \( \ell_{pz} \ll \ell \) as considered in this work, crack paths can be described using a LEFM based theory as the path equation \( (5) \) and follow a random walk with exponent \( \zeta = 0.50 \) (ii) while for materials with \( \ell_{pz} \gg \ell \), toughening is dominated essentially by damage nucleation, growth and coalescence, according to the mechanism proposed in Ref. \([24, 30]\), leading to persistent fracture profiles with \( \zeta \approx 0.6 - 0.7 \).

In conclusion, we brought evidence that cracks in 2D disordered materials with a granular structure follow a directed random walk, and explained theoretically this behavior using a stochastic model based on Fracture Me-
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[31] “See the supplementary material for a detailed description of the experimental and numerical fracture tests, the calculation of the structural lengths $\{\phi_1, \phi_2\}$ and the resolution of the path equation.”.
[38] “Even though one considers specimens under macroscopic tensile loading conditions, the geometrical perturbations of the crack profiles result in some non-zero local shearing in the crack tip vicinity.”.
FIG. 5. Experimental setup: (a) Shematic representation of the fracture test. (b) Snapshot of the pre-notched specimen and the loading device.

I. SUPPLEMENTARY MATERIALS

A. Description of the experimental fracture tests

The fracture experiments are performed using the so-called double torsion test, that is schemically represented in Fig. 5. The total sample size $W \times L$ is $30 \times 60$ cm$^2$ and the sample thickness is $d = 1.5$ cm. This geometry is classically used to achieve slow and controlled mode I crack propagation under tensile loading conditions in thin specimens [41], and has been largely used in rock mechanics [42]. In general, a groove is required to guide the crack parallel to the initial notch of length $c_0 = 10$ cm. However, straight crack propagation could be achieved without it by choosing properly the location of the application of the forces: Two point forces are applied from the top of the specimen on each side of the notch at a distance $w_2 = 2.5$ cm from it and two parallel rails support the specimen from the lower side at a distance $w_1 = 10$ cm from the notch. To avoid indenting the specimen, the upper jaws are not directly in contact with the supper face of the sample, but apply a distributed force over an area of $\simeq 5$ cm$^2$ thanks to a thin plate placed between the jaw and the specimen, as shown on the picture of the experiment in Fig. 5. The upper jaw is displaced vertically at a constant velocity $v_m = 0.1$ mm/s, leading to a slow crack propagation until full failure of the specimen. During a test, the crack propagates over a total distance $\Delta L_{tot} = 50$ cm that corresponds to about 250 bead radius, allowing a rather extended domain of length scale to investigate the scaling properties of the crack path (see Fig. 2).

B. Expression of the structural length scales

{$\{L_1, L_2\}$}

The structural length scales involved in the path equation [7] are specific to the specimen geometry and the loading conditions and follow

$$\begin{align*}
L_1 &= \frac{\pi}{8} \left( \frac{K_I^{(0)}}{T^{(0)}} \right)^2 \\
L_2 &= \frac{1}{\sqrt{2\pi}} \frac{K_I^{(0)}}{A_I^{(0)}}
\end{align*}$$

(7)

where the superscript $(0)$ refers to the fracture test configuration with straight crack. Interestingly, the test geometries used in this study lead to a stress intensity factor $K_I^{(0)}$ that does not depend on the crack length, so that both $L_1$ and $L_2$ are actual constants. But in the general case, their value may vary during the fracturing process.

For the experimental fracture tests represented in Fig. 5 the stress intensity factor at the tip of a straight crack has been calculated by [41] and follows $K_I^{(0)} \simeq (w_1 - w_2)P/d^2\sqrt{W}$. The $T$-stress can be obtained using the relation $T^{(0)} = \sigma_{xx}^{(nc)} - \sigma_{yy}^{(nc)}$ [39] where the superscript $(nc)$ refers to the stress field calculated for the same geometry and loading conditions, but without crack. Since the conditions are close to pure bending in the middle part of the specimen, the stress $\sigma_{xx}^{(nc)} \simeq 0$ along the propagation direction is close to zero while the transverse stress $\sigma_{yy}^{(nc)}$ can be calculated at the bottom surface of the specimen where the tensile state of stress drives the crack, leading to $T^{(0)} \simeq \sigma_{yy}^{(nc)} \simeq -(w_1 - w_2)P/L^2$, so that $L_2^{exp} \simeq \frac{L^2}{W}$ from Eq. (7). To calculate $L_2^{exp}$, we take inspiration from other fracture tests, like the thin stip geometry used in the simulations and analyzed in the next paragraph or the Double Cleavage Drilled Compression test [43] for which the third order term proportional to $A_1 \sim 1/L_2$ in the Williams’ expansion of the mechanical fields at the tip vicinity is set by the specimen width $W$. We assume that this result holds also for the bending test used in this study, leading to $L_2^{exp} \simeq W$. This relation is supported by the physical intuition that the term $h(x)/L_2^{exp}$ in the path equation [5] should not be neglected when the crack “feels” the specimen boundary for excursions $h$ from the straight trajectory of the same order as the specimen width.

For the simulations, the invariance of the thin strip geometry with the propagation direction $x$ under the assumption of a straight crack path makes its analysis easier: The specimen width $W$ is the only relevant length scale, and naturally, one expects it to set both $L_2^{sim}$ and $L_2^{exp}$. The expressions of the structural length scales and their value expressed in number of the bead diameters $\ell$ are summarized here:

$$\begin{align*}
L_1^{exp} &\simeq \frac{L^2}{W} \simeq 600 \ell \quad \text{and} \quad L_2^{exp} \simeq W \simeq 150 \ell \\
L_1^{sim} &\simeq W \simeq 200 \ell \quad \text{and} \quad L_2^{sim} \simeq W \simeq 200 \ell.
\end{align*}$$

(8)
These values will be used in the next section to solve the path equation (5).

C. Perturbative resolution of the path equation

We introduce the two quantities
\[ \epsilon_1 = \sqrt{I/L_1} \quad \text{and} \quad \epsilon_2 = \ell/L_2 \]
that are used to rewrite the path equation (5) as
\[
\frac{dh}{dx} \bigg|_{x^+} = \eta(x) - \epsilon_1 \int_{x-\Delta c}^{x} \frac{h'(u)}{\sqrt{\ell(x-u)}} \, du + \epsilon_2 \frac{h(x)}{\ell}. \quad (10)
\]
Note that the integral along the crack trajectory has been separated into two parts: The straight notch domain \( u < x - \Delta c \), where \( h'(u) = 0 \), that does not contribute to the integral, and the rough section \( x - \Delta c \leq u \leq x \) of the crack path that extends over a length \( \Delta c \), as shown in Fig 5. The values of the structural length scales \( L_1 \) and \( L_2 \) summarized in Eqs. (8) suggest that \( \epsilon_1 \) and \( \epsilon_2 \) can be used as small parameters to solve perturbatively the path equation in the context of the fracture tests performed in this work. As a result, we seek for a solution in the form
\[
h(x) = h^{(0)}(x) + \epsilon_1 h_1(x) + \epsilon_2 h_2(x). \quad (11)
\]
Inserting this form of \( h(x) \) in the path equation (10) and separating zeroth order terms from the ones proportional to \( \epsilon_1 \) or \( \epsilon_2 \) lead to two new equations
\[
\begin{align*}
\frac{dh^{(0)}}{dx} \bigg|_{x^+} &= \eta(x) \\
\epsilon_1 \frac{dh_1}{dx} + \epsilon_2 \frac{dh_2}{dx} &= -\epsilon_1 I(x) + \epsilon_2 \frac{h^{(0)}(x)}{\ell}
\end{align*} \quad (12)
\]
where we note
\[
I(x) = \int_{x-\Delta c}^{x} \frac{h^{(0)}(u)}{\sqrt{\ell(x-u)}} \, du. \quad (13)
\]
We remind that the term \( \eta \) describes the local shear perturbations resulting from the material microstructure, so we expect it to behave as a short range correlated quenched noise. As a result, the solution \( h^{(0)} \) of the zeroth order equation given in Eq. (12) is a directed random walk that is consistent with the numerical and experimental observations reported in this study. In the following, we seek to determine the contribution of the other terms proportional to \( \epsilon_1 \) and \( \epsilon_2 \).

In particular, we are interested in the correlation function of the local slopes along the crack path that, using the decomposition (11), follows
\[
C(\delta x) = \langle h'(x)h'(x+\delta x) \rangle_x \\
= \langle h^{(0)}(x)h^{(0)}(x+\delta x) \rangle_x \\
+ \langle h^{(0)}(x) \left[ \epsilon_1 h_1'(x+\delta x) + \epsilon_2 h_2'(x+\delta x) \right] \rangle_x \\
+ \langle h^{(0)}(x+\delta x) \left[ \epsilon_1 h_1'(x) + \epsilon_2 h_2'(x) \right] \rangle_x + \ldots 
\]
where the terms proportional to \( \epsilon_1^2 \) and \( \epsilon_2^2 \) have been neglected. We now use the expressions (12) of the zero and first order contributions to the total slope: The first line coincides with the correlator \( C_\eta(\delta x) = \langle \eta(x)\eta(x+\delta x) \rangle_x \) of the quenched noise that is essentially zero for \( \delta x > \ell \). Then, we notice that the term \( \epsilon_1 h^{(0)}(x+\delta x) = \eta(x+\delta x) \) has no correlation with the other terms \( h_1'(x) \) and \( h_2'(x) \) that depends on the value of \( h^{(0)}(x) \), and so on the values of \( \eta(\bar{x}) \), in \( \bar{x} \leq x \). So finally, injecting the expressions (12) into the Eq. (14) leads to
\[
C(\delta x) = C_\eta(\delta x) - \epsilon_1 \langle \eta(x)I(x+\delta x) \rangle_x \\
+ \epsilon_2 / \ell \langle \eta(x)h^{(0)}(x+\delta x) \rangle_x
\]
(15)
To calculate the integral \( I(x) \), we discretize the crack path \( \{h_k^{(0)}\}_{-N \leq k \leq n} \) over the domain \( [x - \Delta c \ x + \delta x] \) using \( N + n \) steps of length \( \ell \) where \( N = \Delta c / \ell \) and \( n = \delta x / \ell \). This assumes that \( \eta \) remains nearly constant over distances \( \ell \), that corresponds to approximate the crack trajectory \( h^{(0)} \) as a succession of straight segments with slopes \( \eta_k = h^{(0)}_{k+1} - h^{(0)}_k \ell \) where \( h^{(0)}_k = h^{(0)}(x + k\ell) \).

This approximation is consistent with the behavior of the crack in the simulations that follows straight grain boundaries of typical length \( \ell \) while is describe reasonably well the crack trajectory observed in the experiments. This discretization leads to the expressions
\[
\begin{align*}
I(x+\delta x) &= \sum_{k=-N}^{n-2} \frac{\eta_k}{\sqrt{n-k}} + 2\eta_{n-1} \\
h^{(0)}(x+\delta x) &= \ell \sum_{k=-N}^{n-1} \eta_k
\end{align*} \quad (16)
\]
Since the \( \{\eta_k\}_{-N \leq k \leq n} \) are uncorrelated random variables, \( \langle \eta_i \eta_j \rangle = 0 \) if \( i \neq j \) and \( \langle \eta_i^2 \rangle = \sigma_i^2 \) where \( \sigma_i \) is the amplitude of the quenched noise describing the material microstructure. Using the expressions (16) in Eq. (15), one obtains the correlations of the slopes along the crack path
\[
C(\delta x) = C_\eta(\delta x) - \epsilon_1 \sigma_1^2 \sqrt{\frac{\ell}{\delta x}} + \epsilon_2 \sigma_2^2 \quad (17)
\]
that writes as the correlator \( C_\eta \) of the quenched noise, plus some corrections emerging from the terms proportional to \( \epsilon_1 \) and \( \epsilon_2 \) in the original path equation. Since the noise correlator is of order \( \sigma_i^2 = C_\eta(0) \), these corrections proportional to \( \epsilon_1 \) and \( \epsilon_2 \) are small.

In this study, the correlation \( C(\delta x) \) of the local crack slopes, and so \( C_\eta(\delta x) \), was observed to decay exponentially fast. As a result, the corrections provided in Eq. (17) might become relevant at large distances, typically for \( \delta x \approx 10 \ell \) using the values of \( L_1 \) and \( L_2 \) given in Eq. (8) for the configurations explored in this work. However, the correlation function \( C(\delta x) \) is then very small, of the order of a few percents, that is of the same order as the statistical uncertainty resulting from the finite size.

\[ \text{(5)} \]

\[ \text{(8)} \]

\[ \text{(9)} \]

\[ \text{(10)} \]

\[ \text{(11)} \]

\[ \text{(12)} \]

\[ \text{(13)} \]

\[ \text{(14)} \]

\[ \text{(15)} \]

\[ \text{(16)} \]

\[ \text{(17)} \]
of our sampling. As a result, we were not able to detect any correction to the exponential decay of the slope correlations both in the experiments and in the simulations.

Finally, it is worth noting that the integral acting as a memory term in the path equation produces long range correlations in the local slopes that decay as $\sim \delta x^{-1/2}$. It also produces negative correlations of slopes that would lead to an anti-persistent behavior of the crack path if the crack propagation direction was not dominated by the randomness of the microstructure. This relates to the negative sign of the $T$-stress that tends to redirect the crack along the average crack growth direction after out-of-plane excursions. In our experiments and simulations, the parameter $\epsilon_1$ was too small to measure this effect, but its exploration in systems designed for this purpose would certainly be interesting and could be used as a more restrictive test of the crack path equation proposed in this study.

**Simulation part**

Large scale simulations of intergranular failure based on sequential analyses –

These microstructures are embedded in the process region of a notched specimen that is subsequently loaded by imposing uniform tensile stress on both sample faces parallel to the notch. This test geometry ensures crack propagation from the notch through the whole specimen with a straight trajectory on average. The boundary conditions of the test setup as well as the constitutive models for bulk and grain boundary behavior are identical to those reported in Ref. [33].

Intergranular crack paths are computed with a Generalized Finite Element Method for polycrystals [32]. This method provides an accurate description of the stress field in a consolidated granular solid and yields reliable crack paths [33]. However, in order to explore the scaling properties of the corresponding crack profiles, the number of grains that need to be incorporated would be so large that direct numerical simulations are prohibitively expensive. To overcome this difficulty, a sequential analysis approach has been devised. The approach involves the division of a simulation into a suitable number of sub-simulations thus reducing the computational effort without sacrificing accuracy. In each sub-simulations, a process window (each of the regions enclosed by a blue line in Fig. 1(b)) is defined based on the extent of the nonlinear region around the crack tip. This window follows the propagating crack tip and is relocated accordingly. A highly refined mesh, conforming to the procedure defined in Ref. [33], is used around the crack tip. Outside the process window, a coarse mesh with at least two elements along each grain boundary is employed. Grain boundaries ahead of the process zone are given a high stiffness to simulate perfect bond whereas those in the wake of the process zone are given a zero elastic stiffness to simulate a traction-free crack. We have compared our approach with a monolithic approach on a series of granular topologies and the resulting crack paths are identical. With this approach, detailed in Ref. [44], large crack propagation simulations can be carried out by considering a suitable number of cheaper sub-simulations.

$^1$ AS to Erik: reliable in the sense that can be trusted